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Pseudoconvex fully nonlinear partial differential operators: strong comparison theorems ☆

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Abstract

We introduce a new class of curvature PDOs describing relevant properties of real hypersurfaces of $\mathbb{C}^{n+1}$. In our setting, the pseudoconvexity and the Levi form play the same role as the convexity and the real Hessian matrix play in the real Euclidean one. Our curvature operators are second-order fully nonlinear PDOs not elliptic at any point. However, when computed on generalized s-pseudoconvex functions, we shall show that their characteristic form is nonnegative definite with kernel of dimension one. Moreover, we shall show that the missing ellipticity direction can be recovered by taking into account the CR structure of the hypersurfaces. These properties allow us to prove a strong comparison principle, leading to symmetry theorems for domains with constant curvatures and to identification results for domains with comparable curvatures.

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1. Introduction

In this paper, we are concerned with some notions of curvatures associated with pseudoconvexity and Levi form the way, for instance, the classical Gauss and Mean curvatures are related to the convexity and to the Hessian matrix.

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The curvature operators, we shall deal with lead to a new class of second-order fully nonlinear equations whose *characteristic form*, when computed on *generalized pseudoconvex functions*, are nonnegative definite with *kernel of dimension one*. Then the relevant equations are not elliptic at any point. However, we shall show that they have the following redeeming feature: the missing ellipticity direction can be recovered by suitable commutation relations. We shall use this property to prove a strong comparison principle leading to some symmetry theorems for domains with constant curvatures, and identification results for domains with comparable curvatures.

A class of equations strongly related to the ones studied in this paper were first introduced in [11,7].

Let $bD = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ be a real manifold, boundary of the domain $D = \{z \in \mathbb{C}^{n+1} : f(z) < 0\}$. We assume f is a real-value function with continuous second-order derivatives and such that

$$\partial_p f := (f_{z_1}(p), \dots, f_{z_{n+1}}(p)) \neq 0$$

at any point $p \in bD$. Hereafter, we shall use the notations

$$f_{z_\ell} = \frac{\partial f}{\partial z_\ell}, \quad f_{\bar{z}_\ell} = \frac{\partial f}{\partial \bar{z}_\ell}.$$

We shall also write f_ℓ and $f_{\bar{\ell}}$ instead of f_{z_ℓ} and $f_{\bar{z}_\ell}$, respectively. Similar notations will be used for the second-order derivatives. Let us denote by $T_p^{\mathbb{C}}(bD)$ the complex tangent space to bD at the point p:

$$T_p^{\mathbb{C}}(bD) = \{h \in \mathbb{C}^{n+1} : \langle h, \bar{\partial}_p f \rangle = 0\}$$

with $\langle \cdot, \cdot \rangle$ the usual Hermitian inner product in $\mathbb{C}^{n+1}$ and $\bar{\partial}_p f = (f_{\bar{z}_1}(p), \dots, f_{\bar{z}_{n+1}}(p))$.

Let us denote by $\mathscr{H}_p(f)$ the complex Hessian matrix of the function f at p,

$$\mathscr{H}_p(f) := (f_{j,\bar{k}}(p))_{j,k=1,\dots,n+1}.$$

The Hermitian form

$$\zeta \mapsto L_p(f, \zeta) := \langle \mathscr{H}_p^T(f)\zeta, \zeta \rangle = \sum_{j,k=1}^{n+1} f_{j,\bar{k}} \zeta_j \bar{\zeta}_k,$$

when restricted to $T_p^{\mathbb{C}}(bD)$, is called the Levi form of the function f at the point p. It is a standard fact that $L_p(f, \cdot)/T_p^{\mathbb{C}}(bD)$ is the biholomorphic invariant part of the real Hessian form of f. One way to derive it is to look for a biholomorphic invariant analogue of the Euclidean convexity (see e.g. [6,9]). The Levi form plays a crucial role in the study of the envelopes of holomorphy, in geometric theory of several complex variables: for details on this topic, we refer the reader to [6,9,5,10–12].

We remind that D is called strictly Levi-pseudoconvex if the Levi form of f is strictly positive definite at any point of bD.

Let $B = \{u_1, \dots, u_n\}$ be an orthonormal basis of $T_p^{\mathbb{C}}(bD)$. We call *B-normalized Levi matrix* of bD at a point $p \in bD$ the $n \times n$ Hermitian matrix

$$L_p(f, B) = \left(\frac{1}{|\partial_p f|} \langle \mathscr{H}_p^T(f) u_j, u_k \rangle \right)_{j,k=1,\dots,n}. \tag{1}$$

Obviously, $L_p(f, B)$ depends on the defining function f and on the basis B of $T_p^{\mathbb{C}}(bD)$. However, its eigenvalues only depend on the domain D. More precisely, if f' and B' are respectively another defining function of D and another orthonormal basis of $T_p^{\mathbb{C}}(bD)$ then

$$\lambda(L_p(f, B)) = \lambda(L_p(f', B'))$$

(see the Appendix, Proposition A.1).

We agree to denote $\lambda(A)$ the set of eigenvalues of a $n \times n$ Hermitian matrix A. In what follows, for brevity of notations, we shall write $\lambda_p(bD)$ instead of $\lambda(L_p(f, B))$.

Following the paper [3], where classical real Hessian matrices are considered, we give the following definition.

Definition 1.1. We call generalized symmetric function in $\mathbb{R}^n$ an application $s : \Sigma \to \mathbb{R}$, with $\Sigma \subseteq \mathbb{R}^n$ and such that

(i) Σ and s are invariant with respect to one-to-one rearrangements of $\lambda_1, \dots, \lambda_n$. Moreover, Σ is an open cone contained in the half-space

$$\left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sum_{j=1}^n \lambda_j > 0 \right\}$$

and if $\lambda(A), \lambda(B) \in \Sigma$, then $\lambda(\theta A + (1-\theta)B) \in \Sigma$ for every $\theta \in [0, 1]$;

(ii) s is smooth and

$$\frac{\partial s}{\partial \lambda_j}(\lambda) > 0 \quad \forall \lambda \in \Sigma \quad \forall j = 1, \dots, n;$$

(iii) the function $A \to S(A)$, defined by

$$S(A) = s(\lambda(A)),$$

is smooth and $S(A) \to 0$ as $A \to 0$.

Given a *generalized symmetric function*

$$s : \Sigma \to \mathbb{R}, \quad \Sigma \subseteq \mathbb{R}^n, \tag{2}$$

the real-value map

$$p \mapsto s(\lambda_p(bD)), \quad p \in bD$$

can be seen as a *geometric feature* of bD.

The domain D will be called *s-admissible* if

$$\lambda_p(bD) \subseteq \Sigma \quad \forall p \in bD.$$

We call *s-pseudoconvex* an s-admissible domain D such that

$$s(\lambda_p(bD)) > 0 \quad \forall p \in bD.$$

A defining function f of a domain D will be called *s-admissible* if D is s-admissible. Finally, the real number

$$S_p(bD) \coloneqq s(\lambda_p(bD)) \tag{3}$$

will be called the *s-pseudocurvature* of bD at p.

We explicitly remark that the notions of s-pseudoconvexity and of s-pseudocurvature are independent of the particular choice of the defining function of D.

Example 1.1 (Total Levi-curvature). Let $\Sigma = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_j > 0 \quad \forall j = 1, \ldots, n\}$ and

$$s : \Sigma \to \mathbb{R}, \quad s(\lambda_1, \ldots, \lambda_n) = \prod_{j=1}^{n} \lambda_j. \tag{4}$$

It is quite obvious that this function satisfies conditions (i) and (ii). It also satisfies (iii) because $S(A) = \det A$.

A domain D is strictly Levi-pseudoconvex iff it is s-pseudoconvex, with s given by (4). In this case the s-pseudocurvature will be simply called *total Levi-curvature* and will be denoted by $K_p^{(n)}(bD)$. This curvature can be seen as the pseudoconvex counterpart of the Gauss curvature for boundaries of domains in real Euclidean spaces.

The notion of total Levi-curvature was implicitly introduced in the papers [1,11]: it was explicitly written in [7,8]. By using the definition given in [1] one realizes that

$$K_p^{(n)}(bD) := -\frac{1}{|\partial_p f|^{n+2}} \det \begin{pmatrix} 0 & f_{\bar{1}} & \cdots & f_{\overline{n+1}} \\ f_1 & f_{1,\bar{1}} & \cdots & f_{1,\overline{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+1} & f_{n+1,\bar{1}} & \cdots & f_{n+1,\overline{n+1}} \end{pmatrix}. \tag{5}$$

With this formula in hands it is easy to compute the total Levi-curvature of the sphere of radius R, boundary of the ball

$$D_R = \{z \in \mathbb{C}^{n+1} : |z|^2 < R^2\}.$$

We have

$$K_p^{(n)}(bD_R) = \left(\frac{1}{R}\right)^n \quad \forall p \in bD_R.$$

If we consider the cylinder

$$C_R = \left\{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{j=1}^{n} |z_j|^2 < R^2\right\}$$

from (5) we readily get

$$K_p^{(n)}(bC_R) = 0 \quad \forall p \in bC_R.$$

We would like to notice the existence of cylinder-like domains whose boundaries have strictly positive total Levi-curvature.

For example, if we take

$$C_R^* = \{(z_1, \dots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} \left(\frac{z_j + \bar{z}_j}{2}\right)^2 < R^2\},$$

we have

$$K_p^{(n)}(bC_R^*) = \left(\frac{1}{2R}\right)^n, \quad \forall p \in bC_R^*.$$

Example 1.2 (Classical elementary symmetric functions). Let $j \in \{1, \dots, n\}$ and consider the jth elementary symmetric function

$$\sigma^{(j)}(\lambda_1, \dots, \lambda_n) := \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} \lambda_{i_1} \cdots \lambda_{i_j}.$$

For every $q \in \{1, \dots, n\}$ define

$$\Sigma^{(q)} = \{\lambda \in \mathbb{R}^n : \sigma^{(j)}(\lambda) > 0 \quad \forall j = 1, \dots, q\}$$

and

$$s^{(q)} : \Sigma_q \to \mathbb{R}, \quad s^{(q)}(\lambda) = \frac{\sigma^{(q)}(\lambda)}{\binom{n}{q}}.$$

We know that $s^{(q)}$ satisfies conditions (i)–(iii) (see e.g. [3]).

We may simply call *q-pseudoconvex* a domain which is $s^{(q)}$-pseudoconvex. We shall also denote by $K_p^{(q)}(bD)$ the $s^{(q)}$-pseudocurvature of bD at the point p. Obviously, when $q = n$, $K_p^{(q)}$ is the total Levi-curvature of the previous Example 1.1. When $q = 1$, $K_p^{(q)}$ might be called the *mean Levi curvature* of bD. Indeed

$$s^{(1)}(\lambda) = \frac{\lambda_1 + \cdots + \lambda_n}{n}.$$

Just to show an example, we want to compute the $K^{(q)}$-curvature of the boundary of the ball D_R. Since $f(z) = |z|^2 - R^2$ is a defining function for D_R, we have

$$L_p(f, B) = \frac{1}{R} I_n \quad \forall p \in bD_R,$$

for any orthonormal basis B of the complex tangent space. Then, all the eigenvalues of the normalized Levi form are equal to $\frac{1}{R}$, so that

$$K_p^{(q)}(bD_R) = \left(\frac{1}{R}\right)^q \quad \forall p \in bD_R. \tag{6}$$

Classical elementary symmetric functions of the eigenvalues of the Levi form were considered by Bedford and Gaveau in [1]. Taking into account the definitions given in that paper, we can recognize that

$$K_p^{(q)}(bD) = -\frac{1}{\binom{n}{q}} \frac{1}{|\partial_p f|^{q+2}} \sum_{1 \leqslant i_1 < \cdots < i_{q+1} \leqslant n+1} \Delta_{(i_1, \cdots, i_{q+1})}(f),$$

where

$$\Delta_{(i_1,\cdots,i_{q+1})}(f) = \det\begin{pmatrix} 0 & f_{\bar{i}_1} & \cdots & f_{\bar{i}_{q+1}} \\ f_{i_1} & f_{i_1,\bar{i}_1} & \cdots & f_{1,\bar{i}_{q+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_{q+1}} & f_{i_{q+1},\bar{i}_1} & \cdots & f_{i_{q+1},\bar{i}_{q+1}} \end{pmatrix}.$$

We want to remark that all the previous definitions can be "localized" in a quite obvious way. Then, we can extend the notion of s-pseudoconvexity to the graphs of functions defined in an open subset of $\mathbb{R}^{2n+1}$.

Let Ω be an open subset of $\mathbb{R}^{2n+1}$ and let $u \in C^2(\Omega, \mathbb{R})$. Denote by

$$\Gamma(u) := \{(\xi, \tau) \in \Omega \times \mathbb{R} : u(\xi) < \tau\}$$

and by

$$\gamma(u) := \{(\xi, u(\xi)) : \xi \in \Omega\}$$

the epigraph and the graph of u respectively. With the usual identification of $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$ we shall consider $\Gamma(u)$ and $\gamma(u)$ as subsets of $\mathbb{C}^{n+1}$. We say that u is s-pseudoconvex if $\Gamma(u)$ is s-pseudoconvex at any point of $\gamma(u)$.

As we shall prove in next section, if u is s-pseudoconvex and $\gamma(u)$ has a prescribed s-pseudocurvature, then u satisfies a fully nonlinear equation for which, in Section 4, we shall prove a strong comparison principle. From this result we straightforwardly obtain the following Theorem 1.1, the main application we show of our comparison principle.

Theorem 1.1. *Let D and D' be s-pseudoconvex domains of $\mathbb{C}^{n+1}$ with connected boundaries. Suppose the following conditions are satisfied*:

(a) $D' \subseteq D$ *and* $bD \cap bD' \neq \emptyset$,
(b) $S_{p'}(bD') \leqslant S_p(bD)$ *for every* $p \in bD$ *and* $p' \in bD'$.

Then $D' = D$.

We shall prove this theorem at the end of Section 5. From Theorem 1.1 we easily get the following corollaries:

Corollary 1.1. *Let $D \subseteq \mathbb{C}^{n+1}$ a q-pseudoconvex domain with connected boundary, $1 \leqslant q \leqslant n$. Assume there exists a ball $D_R(z_0) \subseteq D$ tangent to bD at some point of bD. Then, if*

$$K_p^{(q)}(bD) \geqslant \left(\frac{1}{R}\right)^q \quad \forall p \in bD,$$

we have $D = D_R(z_0)$.

Proof. It follows from Theorem 1.1 and identity (6). $\square$

Corollary 1.2. *Let $u : \mathbb{R}^{2n+1} \to \mathbb{R}$ be a C^2 and q-pseudoconvex function. Denote by $K^{(q)}(\xi, u)$ the $K^{(q)}$-curvature of the graph of u at the point $(\xi, u(\xi))$. Then*

$$\inf_{\xi \in \mathbb{R}^{2n+1}} K^{(q)}(\xi, u) = 0. \tag{7}$$

Proof. Since u is q-pseudoconvex, it is $K^{(q)}(\xi, u) > 0$ for every $\xi \in \mathbb{R}^{2n+1}$. By contradiction assume

$$m := \inf_{\xi \in \mathbb{R}^{2n+1}} K^{(q)}(\xi, u) > 0.$$

There exists a ball $D_{1/m}(z_0)$ which is interior tangent to the epigraph of u at some point $p_0 = (\xi_0, u(\xi_0))$. Then, by Corollary 1.1, $\Gamma(u) = D_{1/m}(z_0)$. This is absurd because $\Gamma(u)$, the epigraph of u, is unbounded. $\square$

A stronger result, generalizing a theorem by Bedford and Gaveau [1], can be obtained by directly using the comparison principle of Theorem 5.1 in Section 5.

Corollary 1.3. *Let $u : B_R \to \mathbb{R}$ be a C^2 q-pseudoconvex function in the ball of $\mathbb{R}^{2n+1}$*

$$B_R := \{\xi \in \mathbb{R}^{2n+1} : |\xi| < R\}.$$

Then

$$R \leqslant \sup_{\xi \in B_R} \left(\frac{1}{K^{(q)}(\xi, u)} \right)^{1/q}. \tag{8}$$

Proof. Suppose (8) is false. Then, there exists $r > 0$, $r < R$, such that

$$K^{(q)}(\xi, u) > \left(\frac{1}{r}\right)^q \quad \forall \xi \in B_r. \tag{9}$$

On the other hand, there exists a ball $D_r(\alpha)$ of $\mathbb{R}^{2n+2}$, with radius r and center at α, contained in $\Gamma(u)$ and touching $\gamma(u)$ at a point $p_0 = (\xi_0, u(\xi_0))$.

Let us consider the function $v : B_r(\beta) \to \mathbb{R}$ whose graph $\gamma(v)$ is the lower hemisphere of $bD_r(\alpha)$. Then, by (9) and (6)

$$K^{(q)}(\xi, u) > K^{(q)}(\xi, v) \quad \forall \xi \in B_r(\beta).$$

Moreover, $u \leqslant v$ in $B_r(\beta)$ and $u(\xi_0) = v(\xi_0)$. By Theorem 5.1 it follows that $u \equiv v$ in $B_r(\beta)$, a contradiction because the gradient of u is bounded in $B_r(\beta)$ while that of v is not. $\square$

2. Structure of curvature operators

In this section, we shall show some noteworthy identities and some crucial properties of curvature operators. As in the Introduction, we denote by $D = \{z \in \mathbb{C}^{n+1} | f(z) < 0\}$ a domain of $\mathbb{C}^{n+1}$ whose defining function f is of class C^2 and such that $\partial_p f \neq 0$ when $f(p) = 0$. Then $bD = \{z \in \mathbb{C}^{n+1} | f(z) = 0\}$.

Our first aim is to show an explicit basis of $T_p^{\mathbb{C}}(bD)$, the complex tangent space to bD at the point p. Since $\partial_p f \neq 0$ we may assume $f_{n+1}(p) \neq 0$. Define

$$h_\ell = e_\ell - \alpha_\ell e_{n+1}, \quad \ell = 1, \dots, n,$$

where $(e_j)_{j=1,\dots,n+1}$ is the canonical basis of $\mathbb{C}^{n+1}$, and

$$\alpha_\ell = \alpha_\ell(p) := \frac{f_\ell(p)}{f_{n+1}(p)}. \tag{10}$$

Since

$$\begin{aligned}\langle h_\ell, \bar{\partial}_p f \rangle &= \sum_{j=1}^{n+1} \langle e_\ell - \alpha_\ell e_{n+1}, f_{\bar{j}}(p) e_j \rangle \\ &= f_\ell(p) - \alpha_\ell f_{n+1}(p) = 0,\end{aligned}$$

then $V = \{h_\ell \mid \ell = 1, \dots, n\}$ is a basis of $T_p^{\mathbb{C}}(bD)$.

In what follows we identify h_ℓ with the first-order complex differential operator

$$Z_\ell = \partial_{z_\ell} - \alpha_\ell \partial_{z_{n+1}}, \quad \ell = 1, \dots, n. \tag{11}$$

Then, at the point $p \in bD$

$$Z_\ell(f) = \langle h_\ell, \bar{\partial}_p f \rangle = 0 \tag{12}$$

for every $\ell = 1, \dots, n$. We shall also put

$$\alpha_{\bar{\ell}} = \bar{\alpha}_\ell \quad \text{and} \quad Z_{\bar{\ell}} = \partial_{\bar{z}_\ell} - \alpha_{\bar{\ell}} \partial_{\bar{z}_{n+1}}.$$

For any $j, k \in \{1, \dots, n\}$ let us define

$$A_{j,\bar{k}} = A_{j,\bar{k}}(p) := \langle \mathscr{H}_p^T(f) h_j, h_k \rangle.$$

Then

$$\begin{aligned}A_{j,\bar{k}} &= \langle \mathscr{H}_p^T(f)(e_j - \alpha_j e_{n+1}), (e_k - \alpha_k e_{n+1}) \rangle \\ &= f_{j,\bar{k}} - \alpha_{\bar{k}} f_{j,\overline{n+1}} - \alpha_j f_{n+1,\bar{k}} + \alpha_j \alpha_{\bar{k}} f_{n+1,\overline{n+1}}.\end{aligned}$$

By replacing at the right-hand side the very definitions of α_j and $\alpha_{\bar{k}}$, we obtain

$$A_{j,\bar{k}} = -\frac{1}{|f_{n+1}|^2}\det\begin{pmatrix} 0 & f_{\bar{k}} & f_{\overline{n+1}} \\ f_j & f_{j,\bar{k}} & f_{j,\overline{n+1}} \\ f_{n+1} & f_{n+1,\bar{k}} & f_{n+1,\overline{n+1}} \end{pmatrix}. \tag{13}$$

The eigenvalues of the normalized Levi form can be written in terms of the matrix

$$A(f) = \left(A_{j,\bar{k}}(f)\right)_{j,k=1,\dots,n}.$$

Indeed

Proposition 2.1. *The eigenvalues of the normalized Levi form of bD at the point $p \in bD$ are the eigenvalues of the matrix*

$$C(f) := \frac{1}{|\partial_p f|}A(f)H(f), \tag{14}$$

where

$$H(f) = I_n - \frac{\alpha \cdot \alpha^*}{1+|\alpha|^2}$$

and $\alpha \cdot \alpha^*$ denotes the product of the matrices $\alpha = (\alpha_1, \dots, \alpha_n)^{\mathrm{T}}$ and $\alpha^* = (\bar{\alpha}_1, \dots, \bar{\alpha}_n)$.

Proof. Let us denote by V the $(n+1) \times n$ matrix whose columns are $h_1, \dots, h_n$:

$$V = [h_1, \dots, h_n], \quad h_\ell = e_\ell + \alpha_\ell e_{n+1}.$$

Then, if $U = [u_1, \dots, u_n]$ is an orthonormal basis of $T_p^{\mathbb{C}}(bD)$, there exists an $n \times n$ matrix N such that

$$V = U\,N^T.$$

Since $A^T(f) = V^*\mathscr{H}_p^T(f)V$, we have $A^T(f) = \bar{N}(U^*\,\mathscr{H}_p^T(f)U)\,N^T$ so that

$$\frac{1}{|\partial_p f|}A(f) = NL_p(f,U)N^*,$$

where $L_p(f,U)$ is the U-normalized Levi matrix (see (1)). As a consequence, the matrix

$$L_p(f,U) = \frac{1}{|\partial_p f|}N^{-1}A(f)(N^*)^{-1} \tag{15}$$

has the same eigenvalues of the matrix

$$\frac{1}{|\partial_p f|} A(f)(NN^*)^{-1}.$$

On the other hand, since U is orthogonal, $(NN^*)^T = \bar{N}U^*U\,N^T = V^*V$. Now, an easy computation shows that

$$(V^*V)^T = I_n + \alpha \cdot \alpha^*$$

while for Sherman–Morrison formula

$$(I_n + \alpha \cdot \alpha^*)^{-1} = I_n - \frac{1}{1 + |\alpha|^2} \alpha \cdot \alpha^*.$$

This completes the proof. $\square$

As a first observation on the Proposition 2.1, we show how the *total* and the *mean* Levi-curvatures can be expressed in terms of the matrix $A(f)$. We have

$$K_p^{(n)}(bD) = \frac{|f_{n+1}|^2}{|\partial_p f|^{n+2}} \det A(f) \tag{16}$$

and

$$K_p^{(1)}(bD) = \frac{1}{|\partial_p f|} \mathrm{trace}\left(\left(I_n - \frac{\alpha \cdot \alpha^*}{1 + |\alpha|^2}\right) A(f)\right). \tag{17}$$

Now, we want to show some identities involving the $A_{j,\bar{k}}$'s and the complex vector fields Z_j's in (11).

By means of elementary direct computations one recognizes that

$$Z_j(\alpha_k) = -\frac{1}{f_{n+1}^3} \det \begin{pmatrix} 0 & f_k & f_{n+1} \\ f_j & f_{j,k} & f_{j,n+1} \\ f_{n+1} & f_{n+1,k} & f_{n+1,n+1} \end{pmatrix} \tag{18}$$

and

$$\begin{aligned} Z_j(\alpha_{\bar{k}}) &= -\frac{1}{|f_{n+1}|^2 f_{\overline{n+1}}} \det \begin{pmatrix} 0 & f_{\bar{k}} & f_{\overline{n+1}} \\ f_j & f_{j,\bar{k}} & f_{j,\overline{n+1}} \\ f_{n+1} & f_{n+1,\bar{k}} & f_{n+1,\overline{n+1}} \end{pmatrix} \\ &= \frac{1}{f_{\overline{n+1}}} A_{j,\bar{k}}. \end{aligned} \tag{19}$$

As a consequence

$$Z_j(\alpha_k) = Z_k(\alpha_j) \tag{20}$$

and

$$Z_{\bar{j}}(\alpha_k) = \overline{Z_j(\alpha_{\bar{k}})} = \frac{1}{f_{n+1}}\overline{A_{j,\bar{k}}} = \frac{1}{f_{n+1}}A_{k,\bar{j}}. \tag{21}$$

With these identities in hand it is easy to show the following proposition:

Proposition 2.2. *For every* $j,k = 1,\dots,n$

1. $[Z_j, Z_k] = 0$,
2. $[Z_j, Z_{\bar{k}}] = A_{j,\bar{k}}(f)T$, *where* $T = \frac{1}{f_{n+1}}\partial_{z_{n+1}} - \frac{1}{f_{\overline{n+1}}}\partial_{z_{\overline{n+1}}}$.

Proof. Since

$$[Z_j, Z_k] = (Z_k(\alpha_j) - Z_j(\alpha_k))\partial_{z_{n+1}},$$

from (20) we immediately get $[Z_j, Z_k] = 0$.

2. Since

$$[Z_j, Z_{\bar{k}}] = \left(Z_{\bar{k}}\alpha_j\right)\partial_{z_{n+1}} - (Z_j\alpha_{\bar{k}})\partial_{z_{\overline{n+1}}},$$

by (19) and (21) we have

$$[Z_j, Z_{\bar{k}}] = \frac{1}{f_{n+1}}A_{j,\bar{k}}\partial_{z_{n+1}} - \frac{1}{f_{\overline{n+1}}}A_{j,\bar{k}}\partial_{z_{\overline{n+1}}} = A_{j,\bar{k}}T. \qquad \square$$

Proposition 2.3. *Let s be a generalized symmetric function. Assume that D is a s-admissible domain. Then the s-pseudocurvature of bD at* $p \in bD$ *can be written as follows*:

$$S_p(bD) = \sum_{j,k=1}^{n} a_{j,\bar{k}}A_{j,\bar{k}},$$

where $a_{j,\bar{k}} = \overline{a_{k,\bar{j}}}$ *smoothly depends on* $\partial_z f, \partial_{\bar{z}} f, \partial_z\partial_{\bar{z}} f$ *and*

$$\sum_{j,k=1}^{n} a_{j,\bar{k}}\zeta_j\bar{\zeta}_k \geqslant m|\zeta|^2 \quad \zeta \in \mathbb{C}^n$$

for a suitable $m = m(p,f) > 0$ *continuously depending on p and f.*

Proof. By the very definition of $S_p(bD)$ we have

$$S_p(bD) = S(L_p(f, B)) = s(\lambda_1, \dots, \lambda_n),$$

where $L_p(f, B)$ is the B normalized Levi matrix and $\lambda_1, \dots, \lambda_n$ are its eigenvalues. Moreover, on the set of Hermitian admissible matrices $C = (c_{\ell,\bar{k}})_{\ell,k=1,\dots,n}$ the function $C \mapsto S(C)$ is smooth. We shall denote by $S_{\ell,\bar{k}}(C)$ the derivative of S with respect to $c_{\ell,\bar{k}}$. Since $L = L_p(f, B)$ is admissible, the same holds for $L + C$, for every Hermitian matrix C nonnegative definite and small enough. For such a matrix we have

$$S(L + C) - S(L) = s(\eta_1, \dots, \eta_n) - s(\lambda_1, \dots, \lambda_n),$$

where $\eta_1, \dots, \eta_n$ are the eigenvalues of $L + C$. since $C \geqslant 0$, then $\eta_j \geqslant \lambda_j$, for any $j \in \{1, \dots, n\}$. Moreover, by Definition 1.1-(ii) $\delta = \delta(L) = \frac{1}{2}\min\{\frac{\partial s}{\partial \lambda_j}(\lambda_1, \dots, \lambda_n) | j = 1, \dots, n\} > 0$. Then, if C is small enough

$$\begin{aligned} S(L + C) - S(L) &= \int_0^1 \frac{d}{d\tau} s(\lambda + \tau(\eta - \lambda))\, d\tau \\ &= \sum_{j=1}^n \int_0^1 \frac{\partial s}{\partial \lambda_j}(\lambda + \tau(\eta - \lambda))\, d\tau (\eta_j - \lambda_j) \\ &\geqslant \delta \sum_{j=1}^n (\eta_j - \lambda_j) = \delta(\mathrm{trace}(L + C) - \mathrm{trace}(L)) \\ &= \delta \ \mathrm{trace}\ C. \end{aligned}$$

Let us now apply this inequality to the matrix

$$C = t\zeta \cdot \zeta^*, \quad \zeta \in \mathbb{C}^n$$

and $t > 0$ small enough. We obtain

$$S(L + t\zeta \cdot \zeta^*) - S(L) \geqslant \delta \, \mathrm{trace}\,(C) = \delta t |\zeta|^2. \tag{22}$$

On the other hand

$$\frac{d}{dt} S(L + t\zeta \cdot \zeta^*)|_{t=0} = \sum_{\ell,k=1}^n S_{\ell,\bar{k}}(L) \zeta_\ell \bar{\zeta}_k.$$

Then, from the inequality (22) we get

$$\sum_{\ell,k=1}^n S_{\ell,\bar{k}}(L) \zeta_\ell \bar{\zeta}_k \geqslant \delta |\zeta|^2 \quad \forall \zeta \in \mathbb{C}^n. \tag{23}$$

In what follows we shall denote by ∇S the matrix $(S_{\ell,\bar{k}})_{\ell,k=1,\dots,n}$. Since L is admissible, the same holds for any tL, $0<t\leqslant 1$. Then

$$\begin{aligned} S(L) &= \int_0^1 \frac{d}{dt}S(tL)\,dt = \int_0^1 \operatorname{trace}(\nabla S(tL)\cdot L)\,dt \\ &= (\text{by } (15)) \int_0^1 \operatorname{trace}\left(\nabla S(tL)\frac{1}{|\partial_p f|}N^{-1}A(f)(N^*)^{-1}\right)dt \\ &= \int_0^1 \operatorname{trace}\left(\frac{(N^*)^{-1}\nabla S(tL)N^{-1}}{|\partial_p f|}A(f)\right)dt. \end{aligned}$$

If we denote by $(a_{j,\bar{k}})_{j,k=1,\dots,n}$ the matrix

$$\int_0^1 \frac{(N^*)^{-1}\nabla S(tL)N^{-1}}{|\partial_p f|}\,dt$$

we obtain

$$S(L) = \sum_{j,k=1}^n a_{j,\bar{k}}A_{j,\bar{k}}.$$

On the other hand, by (23)

$$\begin{aligned} \sum_{j,k=1}^n a_{j,\bar{k}}\zeta_j\bar{\zeta}_k &= \int_0^1 \langle \nabla S(tL)N^{-1}\zeta, N^{-1}\zeta\rangle \frac{1}{|\partial_p f|}\,dt \\ &\geqslant \frac{1}{|\partial_p f|}|N^{-1}\zeta|^2\int_0^1 \delta(tL)\,dt \geqslant m|\zeta|^2, \end{aligned}$$

where

$$m := \inf_{|\zeta|=1}\left(\frac{1}{|\partial_p f|}|N^{-1}\zeta|^2\int_0^1 \delta(tL)\,dt\right)$$

is strictly positive and continuously depending on p and on f. $\square$

3. Curvature operators for cartesian surfaces

Let Ω be an open subset of $\mathbb{R}^{2n+1}$ and let $u\in C^2(\Omega,\mathbb{R})$. The aim of this section is to analyze the structure of the curvature operators when applied to the graph of u

$$\gamma(u) := \{(\xi, u(\xi)) : \xi\in\Omega\}.$$

We look at $\gamma(u)$ as (a subset of) the boundary of the domain

$$\Gamma(u) = \{(\xi, \tau) \in \Omega \times \mathbb{R}: u < \tau\}.$$

Let us take as defining function of $\Gamma(u)$

$$f(\xi, \tau) = u(\xi) - \tau. \tag{24}$$

We agree to identify $\mathbb{R}^{2n+1} \times \mathbb{R}$ with $\mathbb{C}^{n+1}$ and to denote by $\xi = (x_1, y_1, \dots, x_n, y_n, t)$ the point of $\mathbb{R}^{2n+1}$. To be consistent, the point of $\mathbb{C}^{n+1}$ will be denoted by $z = (z_1, \dots, z_{n+1})$, with $z_j = x_j + iy_j$, $j = 1, \dots, n$, $z_{n+1} = t + i\tau$.

Reminding the definition given in the Introduction, we say that u is s-admissible if f is s-admissible. The function u is said *s-pseudoconvex* at a point $\xi \in \Omega$ if $\Gamma(u)$ is s-pseudoconvex at the point $(\xi, u(\xi)) \in \gamma(u)$. When u is s-pseudoconvex at any point we simply say that u is s-pseudconvex.

Let $\xi \in \Omega$ and $p = (\xi, u(\xi)) \in \gamma(u)$. With f given as in (24), by (12) we have $0 = Z_\ell(u) - Z_\ell(\tau)$ hence

$$Z_\ell(u) = \frac{i}{2}\alpha_\ell, \tag{25}$$

where

$$\alpha_\ell = \frac{f_\ell}{f_{n+1}} = \frac{\partial_{x_\ell} u - i\partial_{y_\ell}}{\partial_t u + i}. \tag{26}$$

We remark that for function v independent of τ

$$Z_\ell(v) = (\partial_{z_\ell} - \tfrac{1}{2}\alpha_\ell \partial_t)(v),$$

so that $Z_\ell(v) = W_\ell(v)$, where W_ℓ is the complex vector field

$$W_\ell = \partial_{z_\ell} - \tfrac{1}{2}\alpha_\ell \partial_t. \tag{27}$$

We shall denote

$$W_{\bar{\ell}} = \partial_{\bar{z}_\ell} - \tfrac{1}{2}\alpha_{\bar{\ell}} \partial_t.$$

With this notation we can write (25) as follows:

$$W_\ell(u) = \frac{i}{2}\alpha_\ell. \tag{28}$$

From Proposition 2.2 and (19) and (20) we easily obtain the following proposition, in which we also use the notation

$$B_{j,\bar{k}}(u) = A_{j,\bar{k}}(u - \tau). \tag{29}$$

Proposition 3.1. *At any point of Ω we have*

(i) $\frac{1}{2}(W_j W_{\bar{k}} + W_{\bar{k}} W_j)(u) = \frac{B_{j,\bar{k}}(u)}{1+u_t^2}$,

(ii) $[W_j, W_{\bar{k}}] = -4i\frac{B_{j,\bar{k}}(u)}{1+u_t^2}\partial_t$.

Proof.

(i) Identities (19) and (20) imply

$$\begin{aligned} W_{\bar{k}} W_j(u) &= \frac{i}{2} W_{\bar{k}}(\alpha_j) = (\text{since } \alpha_j \text{ does not depend on } \tau) \\ &= \frac{i}{2} Z_{\bar{k}}(\alpha_j) = (\text{by } (20))\, \frac{i}{2} \frac{A_{j,\bar{k}}(f)}{f_{n+1}} \\ &= i\frac{A_{j,\bar{k}}(u-\tau)}{(\partial_t - i\partial_\tau)(u-\tau)} = i\frac{B_{j,\bar{k}}(u)}{\partial_t u + i}. \end{aligned}$$

Hence

$$W_j W_{\bar{k}}(u) = \overline{W_{\bar{j}} W_k(u)} = -i\frac{\overline{B_{k,\bar{j}}(u)}}{\partial_t u - i} = -i\frac{B_{j,\bar{k}}(u)}{\partial_t u - i}.$$

Then

$$(W_j W_{\bar{k}} + W_{\bar{k}} W_j)(u) = iB_{j,\bar{k}}(u)\left(\frac{1}{\partial_t u + i} - \frac{1}{\partial_t u - i}\right) = 2\frac{B_{j,\bar{k}}(u)}{1+u_t^2}.$$

(ii) We first notice that

$$[W_j, W_{\bar{k}}] = -(W_j(\alpha_{\bar{k}}) - W_{\bar{k}}(\alpha_j))\partial_t.$$

Then, since

$$W_j(\alpha_{\bar{k}}) = 2iW_j W_{\bar{k}}(u), \quad W_{\bar{k}}(\alpha_j) = -2iW_{\bar{k}} W_j(u),$$

the assertion follows from (i). □

This proposition implies the following crucial corollary.

Corollary 3.1. *Let $u : \Omega \to \mathbb{R}$ be an s-admissible function. Then*

$$\dim \mathrm{Span}_{\mathbb{C}} \{W_j, [W_j, W_{\bar{k}}] : j,k = 1,\dots,n\} = n+1 \tag{30}$$

at any point of Ω.

Proof. Let $\xi \in \Omega$ be fixed and $p = (\xi, u(\xi))$. Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the normalized Levi-form of $\gamma(u)$ at the point p. Then $\lambda_1 + \cdots + \lambda_n > 0$. As a consequence, by Proposition 2.1 the matrix $(B_{j,\bar{k}})_{j,k=1,\ldots,n}$ is not vanishing. Hence $B_{\ell,\bar{m}} \neq 0$ for a suitable pair (ℓ, m) so that, by Proposition 3.1(ii),

$$W_1, \ldots, W_n, [W_\ell, W_{\bar{m}}]$$

are linearly independent in $\mathbb{C}^{n+1}$. $\square$

Remark 3.1. For (30) to hold at $\xi \in \Omega$ we obviously only need the existence of a pair (ℓ, m) such that $B_{\ell,\bar{m}} \neq 0$ at ξ.

Let us now consider a function

$$K : \Omega \times \mathbb{R} \times \mathbb{R}^{2n+1} \to \mathbb{R}.$$

We say that u has the *assigned s-Levi curvature* K in Ω if

$$S_p(\gamma(u)) = K(\xi, u, Du), \quad p = (\xi, u(\xi))$$

for every $\xi \in \Omega$. Here we have denoted by Du the Euclidean gradient of u in $\mathbb{R}^{2n+1}$. By Propositions 2.3 and 3.1 we immediately obtain

Proposition 3.2. *Let $u \in C^2(\Omega, \mathbb{R})$ be an s-admissible function. If u has the assigned s-Levi curvature K in Ω, then it satisfies the equation*

$$\mathscr{L}u = K(\xi, u, Du), \quad \xi \in \Omega,$$

where $\mathscr{L}$ denotes a second-order fully nonlinear operator of the following type:

$$\mathscr{L} = \mathscr{L}_u := \sum_{j,k=1}^{n} b_{j,\bar{k}} \frac{W_j W_{\bar{k}} + W_{\bar{k}} W_j}{2} \tag{31}$$

and $b_{j,\bar{k}} = \overline{b_{k,\bar{j}}} = b_{j,\bar{k}}(Du, D^2u)$ smoothly depends on Du and on the real Hessian matrix D^2u. Moreover, for every compact set $K \subseteq \Omega$ there exists $m > 0$ such that

$$\sum_{j,k=1}^{n} a_{j,\bar{k}}(Du(\xi), D^2u(\xi))\zeta_j \bar{\zeta}_k \geqslant m|\zeta|^2 \quad \forall \zeta \in \mathbb{C}^n$$

and for every $\xi \in K$.

Proof. By Proposition 2.3 we have

$$\sum_{j,k=1}^{n} a_{j,\bar{k}} B_{j,\bar{k}} = K(\xi, u, Du) \quad \text{in } \Omega.$$

Then, by Proposition 3.1(i), we get the assertions with

$$b_{j,\bar{k}} = \frac{a_{j,\bar{k}}}{1+u_t^2}. \qquad \square$$

In next section, we shall prove our main comparison theorem. For this we need to write our curvature operators in *real* form. Let us put

$$X_j = 2\,\mathrm{Re}\,(W_j), \quad Y_j = -2\,\mathrm{Im}\,(W_j), \quad j = 1, \dots, n$$

and

$$a_j = -\mathrm{Re}\,(\alpha_j), \quad b_j = \mathrm{Im}\,(\alpha_j).$$

Then, since $W_j = \partial_{z_j} - \frac{\alpha_j}{2}\partial_t$, *we have*

$$X_j = \partial_{x_j} + a_j\partial_t, \quad Y_j = \partial_{y_j} + b_j\partial_t, \quad j = 1, \dots, n. \tag{32}$$

With the notations just introduced we can write (25) as follows:

$$(X_j - iY_j)(u) = i\alpha_j = -b_j - ia_j,$$

so that

$$X_j(u) = -b_j, \quad Y_j(u) = a_j.$$

It is easy to see that these relations, together with the structure (32) of X_j and Y_j characterize a_j and b_j as follows:

$$a_j = \frac{u_{y_j} - u_{x_j}u_t}{1+u_t^2}, \quad b_j = \frac{-u_{x_j} - u_{y_j}u_t}{1+u_t^2}.$$

One can also show that these identities are consistent with (26). We now consider the matrix $B := (B_{j,\bar{k}})_{j,k=1,\dots,n}$ in (29), and denote

$$B_1 = \mathrm{Re}(B), \quad B_2 = \mathrm{Im}\,(B).$$

Finally, we define the matrix $C = (c_{j,k})$ as the following $2n \times 2n$ block matrix

$$C = \frac{1}{4}\begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix}. \tag{33}$$

Then, if we rename the vector fields X_j and Y_j as follows:

$$X_j = V_j, \quad Y_j = V_{n+j}, \quad j = 1, \dots, n,$$

the curvature operator in (31) takes the form

$$\mathscr{L} = \mathscr{L}_u = \sum_{j,k=1}^{2n} c_{j,k}(\xi) V_j V_k, \quad c_{j,k}(\xi) = c_{j,k}(Du(\xi), D^2 u(\xi)). \tag{34}$$

Moreover, by Proposition 3.3, for every compact set $K \subseteq \Omega$:

$$\sum_{j,k=1}^{2n} c_{j,k}(\xi)\eta_j\eta_k \geqslant \frac{m}{4} \sum_{j=1}^{2n} \eta_j^2, \quad \forall \eta \in \mathbb{R}^{2n} \tag{35}$$

and for every $\xi \in K$. Then, the operator $\mathscr{L} = \mathscr{L}_u$ is "*elliptic*" *only along* $2n$ *linearly independent directions*. Thus, $\mathscr{L}$ is not elliptic at any point. However, the missing ellipticity direction can be recovered by commutation. Indeed, given the structure of the vector fields V_j's, the commutator $[V_j, V_k]$ takes the following form:

$$[V_j, V_k] = v_{j,k}\partial_t,$$

where $v_{j,k}$ is a suitable function in Ω. By Corollary 3.1, for every point $\xi \in \Omega$ there exists a pair (j,k) such that $v_{j,k}(\xi) \neq 0$. Therefore

$$\dim(\mathrm{Span}_{\mathbb{R}}\{V_j, [V_j, V_k] : j,k = 1, \ldots, 2n\}) = 2n+1, \tag{36}$$

at any point of Ω.

This property will be crucial in the proof of our strong maximum and comparison principles.

4. Strong maximum principle for subelliptic operators

In this section we assume $\Omega \subseteq \mathbb{R}^{2n+1}$ an open set and $X_1, \ldots, X_{2n}$ linear C^1 vector fields in Ω such that

$$\dim(\mathrm{Span}\{X_j(\xi), [X_i, X_j](\xi), i,j = 1, \ldots, 2n\}) = 2n+1 \tag{37}$$

for every $\xi \in \Omega$. We consider the following partial differential operator:

$$\mathscr{M} = \sum_{i,j=1}^{2n} \beta_{i,j}(\xi) X_i X_j + \langle \beta, D \rangle + c, \tag{38}$$

where $\beta_{i,j}$, $\beta = (\beta_1, \ldots, \beta_{2n})$ and c are real continuous functions in Ω. We also assume that for every compact set $K \subset \Omega$ there exists $m = m(K) > 0$ such that

$$\sum_{i,j=1}^{2n} \beta_{i,j}(\xi)\eta_i\eta_j \geqslant m|\xi|^2 \quad \forall \xi \in K \quad \forall \eta \in \mathbb{R}^{2n}, \tag{39}$$

where $m>0$ is a suitable constant. In (38) D and $\langle\cdot,\cdot\rangle$ denote, respectively, the gradient operator and the inner product in $\mathbb{R}^{2n+1}$.

Our aim in this section is to prove the following strong maximum principle for $\mathscr{M}$.

Theorem 4.1. *Let $\Omega_0\subseteq\Omega$ be open and connected. Suppose $w\in C^2(\Omega_0,\mathbb{R})$ and such that*

$$\begin{aligned} &\mathscr{M}w\geqslant 0 \quad \text{in } \Omega_0,\\ &w\leqslant 0 \qquad \text{in } \Omega_0. \end{aligned}$$

Then $w<0$ in Ω_0 or $w\equiv 0$ in Ω_0.

For the proof of this theorem we need some lemmas. The first one shows a weak maximum principle for $\mathscr{M}$ on small open sets.

Lemma 4.1. *For every $\xi_0\in\Omega$ there exists an open set $\mathscr{U}_0$ such that $\overline{\mathscr{U}}_0\subset\subset\Omega$, $\xi_0\in\mathscr{U}_0$, with the following property: if $w\in C^2(\mathscr{U})$ satisfies*

$$\begin{cases} \mathscr{M}w\geqslant 0 & \text{in } \mathscr{U}_0,\\ \limsup\limits_{\xi\to\zeta}\ w(\xi)\leqslant 0 & \forall\zeta\in\partial\mathscr{U}_0, \end{cases}$$

then $w\leqslant 0$ in $\mathscr{U}_0$.

Proof. By hypothesis (39), there exists $\mu\in\mathbb{R}^{2n+1}$ such that

$$\sum_{j,k=1}^{2n}\beta_{j,k}(\xi_0)\langle X_j(\xi_0),\mu\rangle\langle X_k(\xi_0),\mu\rangle>0.$$

Then, if we define

$$v(\xi)=M-\exp(\chi\langle\mu,\xi\rangle),$$

an elementary computation shows that we can choose constants $\chi, M>0$ and a neighborhood $\mathscr{U}_0$ of ξ_0 such that

$$\mathscr{M}v<0,\quad \inf_{\mathscr{U}_0}\ v>0.$$

It is well known that the existence of such a barrier function implies the weak maximum principle for $\mathscr{M}$ in $\mathscr{U}_0$. $\square$

The second lemma we need is a kind of Hopf Lemma. Let $w:\Omega_0\to\mathbb{R}$ be the function in Theorem 4.1. Suppose there exists a point $\xi_0\in\Omega_0$ such that $w(\xi_0)=0$. Theorem 4.1 requires to prove that the relatively closed set

$$F=\{\xi\in\Omega_0 : w(\xi)=0\},$$

actually is equal to Ω_0. We argue by contradiction and suppose $F \neq \Omega_0$. Then, since Ω_0 is connected, $\partial F \cap \Omega_0 \neq \emptyset$. A unitary vector $v \in \mathbb{R}^{2n+1}$ will be called an *outer normal* to F at a point $\xi \in \partial F \cap \Omega$ if there exists $r > 0$ such that the Euclidean ball $B_r(\xi + rv)$ centered at $\xi + rv$ and with radius r, satisfies

$$B_r(\xi + rv) \subseteq \Omega_0 \backslash F. \tag{40}$$

We shall denote by F^* the set

$$F^* = \{\xi \in \partial F \cap \Omega_0 : \text{there exists } v \text{ outer normal to } F \text{ at } \xi\}.$$

Since $\partial F \cap \Omega_0 \neq \emptyset$, it is easy to obtain that $F^* \neq \emptyset$.

Lemma 4.2. *At any point $\xi \in F^*$ and for every outer normal v to F at ξ, we have*

$$\langle X_j(\xi), v \rangle = 0 \quad \forall j = 1, \ldots, 2n. \tag{41}$$

Proof. The proof is based on the previous lemma and on quite standard arguments. We directly refer to [2, Proposition 3.1]. □

Let us denote by $\exp(tX_j)(\xi)$ the solution φ with maximal domain of the Cauchy problem

$$\begin{cases} \dot{\varphi} = X_j(\varphi), \\ \varphi(0) = \xi, \quad \xi \in \Omega. \end{cases}$$

By the previous lemma and [2, Theorem 2.1], if $\xi \in F$ then

$$\exp(tX_j)(\xi) \in F \tag{42}$$

for every t in a neighborhood of $t = 0$ and for every $j \in \{1, \ldots, 2n\}$.

Let $i, j \in \{1, \ldots, 2n\}$ and $0 \leqslant t \leqslant \delta$, with $\delta > 0$ small enough. Define

$$\Phi(t) = \exp(-\sqrt{t}X_j)(\exp(-\sqrt{t}X_i)(\exp(\sqrt{t}X_j)(\exp(\sqrt{t}X_i)(\xi)))).$$

It is well known that Φ is differentiable in $[0, \delta]$ and

$$\dot{\Phi}(0) = [X_i, X_j](\xi). \tag{43}$$

On the other hand, by (42) $\Phi(t) \in F$ so that by (40),

$$|\Phi(t) - (\xi + rv)|^2 \geqslant r^2 = |\Phi(0) - (\xi + rv)|^2.$$

As a consequence

$$\frac{d}{dt}|\Phi(t) - (\xi + rv)|\Big|_{t=0} \geqslant 0,$$

i.e. $2r\langle \dot{\Phi}(0), v\rangle \geqslant 0$. Using (43) we get $\langle [X_i, X_j](\xi), v\rangle \geqslant 0$, and exchanging i and j, $\langle [X_j, X_i](\xi), v\rangle \geqslant 0$. Then $\langle [X_i, X_j](\xi), v\rangle = 0$.

We are now ready to conclude the proof of our strong maximum principle.

Proof of Theorem 4.1. By contradiction suppose $w \neq 0$ and $w = 0$ somewhere. Then $\emptyset \neq F \neq \Omega_0$. It follows that $F^* \neq \emptyset$. For what we have just proved

$$\langle X_j(\xi), v\rangle = 0, \quad \langle [X_i, X_j](\xi), v\rangle = 0$$

for every $\xi \in F^*$ and v outer normal to F at ξ, $\forall i, j = 1, \dots, 2n$. From the *rank condition* (37), this implies $v = 0$, a contradiction because $|v| = 1$. $\square$

5. Strong comparison principle for *s*-pseudoconvex functions

In this section, we prove our main comparison Principle and we shall use the same notations of Section 3.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^{2n+1}$ be open and connected. Let $u, v \in C^2(\Omega)$ be real s-pseudoconvex functions. If $u \leqslant v$ in Ω and*

$$\mathscr{L}u - K(\xi, u, Du) \geqslant \mathscr{L}v - K(\xi, v, Dv)$$

in Ω for some smooth function $K : \Omega \times \mathbb{R} \times \mathbb{R}^{2n+1} \to \mathbb{R}$, then $u \equiv v$ or $u < v$ in Ω.

Proof. For brevity of notation we denote by $L(u)$ and $L(v)$ the normalized Levi matrix of u and v, given by replacing in (14) the defining function f with $u - \tau$ and $v - \tau$, respectively. Since u and v are s-admissible functions, by condition (i) in Definition 1.1 we have that the eigenvalues of $\theta L(u) + (1-\theta)L(v)$ belong to Σ for all $0 \leqslant \theta \leqslant 1$.

Let us put $w = u - v$. We shall show that $\mathscr{M}w \geqslant 0$ in Ω, where $\mathscr{M}$ is an operator of the type (38). First of all we have,

$$\begin{aligned}
\mathscr{L}u - \mathscr{L}v = \mathscr{L}_u u - \mathscr{L}_v v &= S(L(u)) - S(L(v)) \\
&= \int_0^1 \frac{d}{d\theta}(S(\theta L(u) + (1-\theta)L(v)))\, d\theta \\
&= \int_0^1 \operatorname{trace}(\nabla S(\theta L(u) + (1-\theta)L(v)) \cdot (L(u) - L(v)))\, d\theta \\
&= \operatorname{trace}(J \cdot (L(u) - L(v))),
\end{aligned} \tag{44}$$

where

$$J := \int_0^1 \nabla S(\theta L(u) + (1-\theta)L(v))\, d\theta$$

is a positive definite matrix by (23). Now, by taking into account (15) we have

$$\mathrm{trace}(J \cdot (L(u) - L(v))) = \mathrm{trace}(\tilde{J} \cdot (B(u) - B(v))) + \langle \beta, Dw \rangle,$$

where β is a continuous function and

$$\tilde{J} = \frac{2}{\sqrt{1+|Du|^2}}(N^{-1}(u-\tau) \cdot J \cdot (N^*)^{-1}(u-\tau))$$

is a positive Hermitian matrix with continuous coefficients. Moreover, by (29) and Proposition 3.1(i), we have

$$\begin{aligned} B(u) - B(v) &= \frac{1}{2}((1+u_t^2)(W_j W_{\bar{k}} + W_{\bar{k}} W_j)u - (1+v_t^2)(W_j W_{\bar{k}} + W_{\bar{k}} W_j)v)_{j,k=1}^n \\ &= \frac{1+u_t^2}{2}((W_j W_{\bar{k}} + W_{\bar{k}} W_j)u - (W_j W_{\bar{k}} + W_{\bar{k}} W_j)v)_{j,k=1}^n \\ &\quad + \text{first-order derivatives of } w. \end{aligned}$$

In order to write this last term as a second-order operator acting on w, we introduce the notation $W_j[u](w) = (\partial_{z_j} - \frac{1}{2}\alpha_j(u)\partial_t)(w)$. Accordingly, $W_j(u) = W_j[u](u)$.

A direct computation shows that

$$\begin{aligned} (W_\ell W_{\bar{p}} + W_{\bar{p}} W_\ell)(u) - (W_\ell W_{\bar{p}} + W_{\bar{p}} W_\ell)(v) &= (W_\ell[u] W_{\bar{p}}[u] + W_{\bar{p}}[u] W_\ell[u])(w) \\ &\quad + \text{first-order derivatives of } w. \end{aligned} \tag{45}$$

Thus, by defining $V_j[u] = 2\mathrm{Re}(W_j[u])$ and $V_{n+j}[u] = -2\mathrm{Im}(W_j[u])$, we can rewrite (44) as follows:

$$\mathscr{L}_u u - \mathscr{L}_v v = \sum_{j,k=1}^{2n} c_{j,k} V_j[u] V_k[u](w) + \text{first-order derivatives of } w, \tag{46}$$

where $C = (c_{j,k})$ is the positive nonsymmetric block matrix

$$C = \frac{1}{4}\begin{pmatrix} \mathrm{Re}\,\tilde{J} & \mathrm{Im}\,\tilde{J} \\ -\mathrm{Im}\,\tilde{J} & \mathrm{Re}\,\tilde{J} \end{pmatrix}.$$

Obviously,

$$K(\xi, u, Du) - K(\xi, v, Dv) = \text{first-order derivatives of } w + c_1 w. \tag{47}$$

Then, with an obvious meaning of $\mathscr{M}$ by (46) and (47) we have

$$\mathscr{L}u - K(\xi, u, Du) - \mathscr{L}v + K(\xi, v, Dv) = \mathscr{M}w.$$

Thus, $\mathscr{M}w \geqslant 0$ in Ω and $w \leqslant 0$ in Ω. The thesis follows from Theorem 4.1. $\square$

Our Theorem 5.1 contains the strong comparison principle proved by Citti [4] for a quasilinear equation, whose principal part is related to the trace of the Levi form in the case $n = 1$.

We close this section by giving the

Proof of Theorem 1.1. For any fixed $p \in bD \cap bD'$ we shall prove the existence of an open set $U \subseteq \mathbb{C}^{n+1}$, with $p \in U$, such that $bD \cap U = bD' \cap U$. Then, from the connectedness of bD and bD', it will follow $bD = bD'$. This, together with the inclusion $D' \subseteq D$, will give $D = D'$.

Let $p \in bD \cap bD'$. Without loss of generality and by using the notations of Section 3, we may assume $p = (\xi_0, \tau_0)$ with $\xi_0 \in \mathbb{R}^{2n+1}$ and $\tau_0 \in \mathbb{R}$, and the existence of an open set $\Omega \subseteq \mathbb{R}^{2n+1}$ and of a connected open set $U \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2n+1} \times \mathbb{R}$ satisfying:

(i) $p \in U$ and $\xi_0 \in \Omega$,

(ii) there exist $u, v \in C^2(\Omega, \mathbb{R})$ such that

$$D \cap U = \Gamma(u) \cap U, \quad bD \cap U = \gamma(u) \cap U,$$

$$D' \cap U = \Gamma(v) \cap U, \quad bD' \cap U = \gamma(v) \cap U.$$

Then, since $D' \subseteq D$ and $p \in bD \cap bD' \cap U$, we have $u \leqslant v$ in Ω and $u(\xi_0) = v(\xi_0)$. Moreover, u and v are s-pseudoconvex and, by hypothesis b,

$$\mathscr{L}u \geqslant \mathscr{L}v \quad \text{in } \Omega.$$

Then, by Theorem 5.1 $u \equiv v$ in Ω. Hence $bD' \cap U = bD \cap U$, and the proof is complete. $\square$

Appendix

In this appendix, by using the notations of the Introduction, we prove the following proposition:

Proposition A.1. *Let f and f' be defining functions of a domain D. Let $p \in bD$ and let*

$$B = \{u_1, \ldots, u_n\}, \quad B' = \{v_1, \ldots, v_n\}$$

be orthonormal basis of $T_p^{\mathbb{C}}(bD)$. Then $\lambda(L_p(f, B)) = \lambda(L_p(f', B'))$.

Proof. Denote by U the $(n+1)\times n$ matrix

$$U = [u_1, \dots, u_n]$$

and by $A(U)$ the matrix $(\langle \mathscr{H}_p^T(f)u_j, u_k\rangle)_{j,k=1,\dots,n}$. We have

$$A^T(U) = U^* \mathscr{H}_p^T(f)U.$$

Let $V = [v_1, \dots, v_n]$ be the matrix related to another orthonormal basis B' of $T_p^{\mathbb{C}}(bD)$. Now, let R be an $n\times n$ orthonormal matrix such that $V = UR$. Then

$$A^T(V) = R^* U^* \mathscr{H}_p^T(f)UR = R^* A^T(U)R,$$

so that $A(V)$ and $A(U)$ have the same eigenvalues.

Since f and f' are defining functions of D, we have $f' = \sigma f$, where σ is C^1 and strictly positive in a neighborhood of p, see [9, Chapter II, Lemma 2.5]. It follows that

$$f'_{\bar{j},k}(p) = \sigma(p)f_{\bar{j},k}(p) + \sigma_{\bar{j}}(p)f_k(p) + \sigma_k(p)f_{\bar{j}}(p).$$

Therefore, for every $\zeta \in T_p^{\mathbb{C}}(bD)$,

$$\begin{aligned}\langle \mathscr{H}_p^T(f')\zeta,\zeta\rangle &= \sigma(p)\langle \mathscr{H}_p^T(f)\zeta,\zeta\rangle + 2\mathrm{Re}(\langle \zeta, \bar{\partial}_p f\rangle\langle \bar{\partial}_p\sigma,\zeta\rangle)\\ &= \sigma(p)\langle \mathscr{H}_p^T(f)\zeta,\zeta\rangle.\end{aligned}$$

On the other hand

$$\partial_p f' = \sigma(p)\partial_p f.$$

Then

$$\frac{1}{|\partial_p f'|}\langle \mathscr{H}_p^T(f')\zeta,\zeta\rangle = \frac{1}{|\partial_p f|}\langle \mathscr{H}_p^T(f)\zeta,\zeta\rangle$$

for every $\zeta \in T_p^{\mathbb{C}}(bD)$. This completes the proof. □

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